Beam Physics Note
TRI-BN-19-09
May 9, 2019

Kapchinskij-Vladimirskij-Sacherer Envelope Equation from the Low Lagrangian

Thomas Planche, Paul M. Jung

TRIUMF

Abstract: In this note we re-derive Sacherer generalised form of the Kapchinskij-Vladimirskij envelope equation, starting from the Low Lagrangian for electrostatic collisionless plasma. For simplicity, we only treat here the one-dimensional case.

4004 Wesbrook Mall, Vancouver, B.C. Canada V6T 2A3 Tel: 604 222-1047 www.triumf.ca

1 Setting up the One-Dimensional Problem

Let's assume that we work with a beam that is very wide in the y direction, very long in the z direction, and where all particles travel with the same $\dot{z}_0$ longitudinal velocity. In this one-dimensional case, the Low Lagrangian [1] for non-relativistic plasma in the electrostatic limit reduces to:

$$L(x,\dot{x},\phi;t) = \iint f(x_0,\dot{x}_0)\left(\frac{m}{2}\dot{x}^2 - q\phi(x,t) - q\psi(x,t)\right)\mathrm{d}x_0\mathrm{d}\dot{x}_0 + \frac{\epsilon_0}{2}\int \phi'(\bar{x},t)^2\mathrm{d}\bar{x}\,. \quad (1)$$

Time t is the independent variable. The variables $\bar{x}$, x_0 and $\dot{x}_0$ are dummy integration variables. The functions $x = x(x_0,\dot{x}_0,t)$ and $\dot{x} = \dot{x}(x_0,\dot{x}_0,t)$ map initial particle coordinates $(x_0,\dot{x}_0)$ to the corresponding coordinates at time t. ϕ is the self-potential and $\phi'(\bar{x},t) = \frac{\partial\phi}{\partial\bar{x}}$. f is the initial plasma density function. In this one-dimensional model particles are actually infinite sheets of charge which leads to m being a surface mass density, and q being a surface charge density.

For simplicity, and without much loss of generality, we choose to represent the effect of external forces using the scalar potential $\psi(x,t)$. [1] Let's also assume that the external focusing force is purely linear:

$$\psi(x,t) = \frac{1}{2}k(t)x^2\,. \quad (2)$$

2 Fixed-Shape Distribution

Let's assume that the shape of the particle distribution, and by extension the shape of ϕ, is time-independent, and that only their size changes with time. This assumption is true, for instance, for a beam with phase-space elliptical symmetry well-matched to a periodic transport system. It is also true in the more artificial case of a K-V beam with purely linear external forces. This assumption leads to:

$$\phi(x,t) = \phi(x,\sigma(t))\,, \quad (3)$$

where $\sigma = \sqrt{\langle x^2\rangle}$, with:

$$\langle x^2\rangle = \iint f(x_0,\dot{x}_0)\big(x(x_0,\dot{x}_0,t)\big)^2\,\mathrm{d}x_0\mathrm{d}\dot{x}_0\,. \quad (4)$$

Following Sacherer [2], we define the beam emittance $\varepsilon(t)$ as:

$$\varepsilon^2 = \langle x^2\rangle\langle\dot{x}^2\rangle - \langle x\dot{x}\rangle^2\,, \quad (5)$$

Noting that[2]:

$$\dot{\sigma} = \frac{\langle x\dot{x}\rangle}{\sigma}\,, \quad (6)$$

the Lagrangian becomes:

$$\boxed{L(\sigma,\dot{\sigma},\phi;t) = \frac{m}{2}\dot{\sigma}^2 - \frac{m}{2}\frac{\varepsilon^2}{\sigma^2} - \frac{q}{2}k\sigma^2 - q\langle\phi(x,\sigma)\rangle + \frac{\epsilon_0}{2}\int \phi'^2(\bar{x},\sigma)\mathrm{d}\bar{x}}\,. \quad (7)$$

[1] the effect of hard-edge magnetic elements can be taken into account by adding a $q\dot{z}_0A_z(x)$ term, which is equivalent to defining $\psi(x,t) = -\dot{z}_0A_z(x,t)$.

[2] since $\dot{\sigma} = \dot{\sqrt{\langle x^2\rangle}} = \frac{\dot{\langle x^2\rangle}}{2\sqrt{\langle x^2\rangle}} = \frac{2}{2}\frac{\langle x\dot{x}\rangle}{\sigma}$. Mind that $\dot{\langle x^2\rangle} \neq \langle\dot{x}^2\rangle$.

Hamilton's principle of stationary action leads to:

$$0 = \frac{\delta S}{\delta \phi}\,, \tag{8}$$

$$\text{and } 0 = \frac{\delta S}{\delta \sigma}\,, \tag{9}$$

where the action $S = \int L dt$. Equation 8 leads to the one-dimensional Poisson equation:

$$\phi'' = -q\frac{h}{\epsilon_0}\,, \tag{10}$$

where h is projection of the charge distribution on the horizontal axis:

$$h(\bar{x}, t) = \iint f(x_0, \dot{x}_0)\delta\big(x(x_0, \dot{x}_0, t) - \bar{x}\big)\, \mathrm{d}x_0 \mathrm{d}\dot{x}_0\,. \tag{11}$$

Equation 9 leads to:

$$\boxed{0 = \ddot{\sigma} - \frac{\varepsilon^2}{\sigma^3} + \frac{qk}{m}\sigma - \frac{q^2}{2m\epsilon_0}\lambda_1}\,, \tag{12}$$

where λ_1 is a dimensionless[3] number given by:

$$\lambda_1 = -\frac{2\epsilon_0}{q^2}\left(q\frac{\partial\langle\phi\rangle}{\partial\sigma} - \frac{\epsilon_0}{2}\int \frac{\partial \phi'^2}{\partial \sigma}\mathrm{d}\bar{x}\right)\,. \tag{13}$$

Equation Eq. (12) has the form of Sacherer generalised form of the Kapchinskij-Vladimirskij envelope equation [2, 3]. Now is λ_1 the same than σ-independent than the one in Ref. [2]?

Injecting Eq. (10) into Eq. (13) I calculated the value of λ_1 for the same set of distribution than used by Sacherer. I find that the values of λ_1 are σ-independent, and are identical to those found by Sacherer [2]! See Table 1. Sacherer's definition of λ_1 must be equivalent to Eq. (13), but I have not yet figured out a proof...

	$h(x,\sigma)$	$-\frac{2\epsilon_0}{q}\phi'(x,\sigma)$	λ_1	$\sqrt{3}\,\lambda_1$
Uniform	$\begin{cases} \frac{1}{2\sigma\sqrt{3}}, & \lvert x\rvert < \sigma\sqrt{3} \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} \frac{x}{\sigma\sqrt{3}}, & \lvert x\rvert < \sigma\sqrt{3} \\ x/\lvert x\rvert & \text{otherwise} \end{cases}$	$\frac{1}{\sqrt{3}}$	1
Parabolic	$\begin{cases} \frac{3(5\sigma^2 - x^2)}{20\sqrt{5}\sigma^3}, & \lvert x\rvert < \sigma\sqrt{5} \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} x\frac{15\sigma^2 - x^2}{10\sqrt{5}\sigma^3}, & \lvert x\rvert < \sigma\sqrt{5} \\ x/\lvert x\rvert & \text{otherwise} \end{cases}$	$\frac{9}{7\sqrt{5}}$	0.996
Normal	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}$	$\operatorname{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)$	$\frac{1}{\sqrt{\pi}}$	0.977
Hollow	$\frac{3}{\sigma^3}\sqrt{\frac{3}{2\pi}}x^2 e^{-\frac{3x^2}{2\sigma^2}}$	$\operatorname{erf}\left(\sqrt{\frac{3}{2}}x/\sigma\right) - \frac{x\sqrt{\frac{6}{\pi}}}{\sigma}e^{-\frac{3x^2}{2\sigma^2}}$	$\frac{7}{4\sqrt{3\pi}}$	0.987

Table 1: Numerical evaluuation of λ_1 using Eq. (13) for the same set of example distributions than used in Ref. [2]. The numbers in the last column are truncated after the third significant digit.

3 Conclusion

We have shown how to re-derive Sacherer's envelope equation from Low's Lagrangian for the 1-D transverse case. We have not tried yet to re-derive the other cases treated by Sacherer: (1) the 1-D longitudinal, (2) the 2-D transverse, or (3) the full 3-D...

[3]mind that q is a surface charge density: $\frac{\mathrm{C}}{\mathrm{m}^2}$.

References

[1] F. E. Low, A Lagrangian formulation of the Boltzmann-Vlasov equation for plasmas, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences 248 (1253) (1958) 282–287.
URL http://www.jstor.org/stable/100602

[2] F. J. Sacherer, RMS envelope equations with space charge, Tech. Rep. DL/70-12 http://cds.cern.ch/record/322516/files/cer-000245740.pdf, CERN (1970).

[3] I. M. Kapchinskij, V. V. Vladimirskij, Limitations Of Proton Beam Current In A Strong Focusing Linear Accelerator Associated With The Beam Space Charge, in: Proceedings, 2nd International Conference on High-Energy Accelerators and Instrumentation, HEACC 1959: CERN, Geneva, Switzerland, September 14-19, 1959, 1959, pp. 274–287.