Beam Physics Note
TRI-BN-21-17
November 4, 2021

Review on Electric Focusing

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Abstract: In preparation of the USPAS course, as a pedagogical exercise, I re-derive Reiser's formula from Baartman's 'LINAC' Hamiltonian. The formula thus obtained is a little more general: it is valid for all gap geometries (by means of an asymmetry parameter ι), depends only on the gap effective length (eliminating the need to evaluate Reiser's geometrical parameter $\frac{F}{b}$), and is relativistic (not that this matters very much in practice).

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In this section we show how to calculate the transverse focusing/defocusing effect that arises from the crossing of an rf gap. We will give a particular attention to the effect of the rf gap geometry, as in a cyclotron one may encounter round(-ish) gaps at injection, and flat (typically horizontal) gaps in the rest of the machine.

If you are familiar with the way linear (hadron) accelerators work, you will know that beams are usually accelerated slightly off the crest of the rf waveform, so that particles arriving late gain slightly more velocity than the particles arriving early. This allows the late particles to catch up with the early ones, and vice-versa, holding the bunch together following a stable longitudinal motion. Each rf gap thus acts as a lens which focuses the beam in the longitudinal direction. Unavoidably, Maxwell's equations – as we will show in this section – dictate that this longitudinal focusing effect goes hand in hand with a transverse defocusing effect, hence the use of quadrupoles or solenoids for transverse focusing in linacs. On the other side of the rf waveform, the effect is opposite: the rf gap defocuses longitudinally and focuses transversely. While this mode of operation is unacceptable for a hadron linac, because of the lack of longitudinal focusing, this is how cyclotrons operate. How can this be?

The phase-dependent energy gain does not lead to any longitudinal defocusing (or focusing) in a cyclotron. Cyclotrons being isochronous cannot debunch. Like in a synchrotron at transition, particles don't catch up or lag behind any further: the longitudinal phase space is 'frozen'.

What is more, the transverse focusing that the rf gap can provide is a blessing for cyclotrons at very low energies. At low energies indeed, when the radius of the orbit is comparable with the height of the magnetic gap, the effect of separated sectors washes off, leading to a very weak variation of the magnetic field with azimuth. And in the absence of azimuthal field dependence: there is no AVF focusing[1]. One has to rely on the vertical focusing from the rf gap to ensure stable vertical motion until the radius of the orbit becomes large enough for the AVF focusing to kick in.

This phase dependent transverse focusing is often referred to, in the context of cyclotrons, as 'electric focusing' [5, 3, 2], hence the title of this section. One last thing to understand, before we dive into a more quantitative description of this effect, is that it depends on the symmetry of the rf gap: a cylindrically symmetric gap will focus the beam as much is the horizontal direction as in the vertical directly. On the contrary, an horizontally 'flat' rf gap will not focus in the horizontal direction: after all, by symmetry, it has no 'gap center' towards which particles may be focused in the horizontal direction. It focuses however particles in the vertical direction (towards the median plane). Maxwell's equations impose that the amount of transverse focusing follows a sum rule: the vertical focusing from a flat rf gap is twice that from a cylindrical gap, as we are about to discover.

0.1 Linear Motion Hamiltonian

Let's start from the s-independent-variable Hamiltonian, choosing a straight reference trajectory ($\frac{1}{\rho} = 0$):

$$\mathcal{H}_s(x, P_x, z, P_z, t, -E) = -P_s = -qA_s - \sqrt{\left(\frac{E - q\Phi}{c}\right)^2 - m^2c^2 - (P_x - qA_x)^2 - (P_z - qA_z)^2} \quad (1)$$

First, we need to choose a suitable set of potentials to represent our problem: along the optical axis the resulting electric field should be an arbitrary function $\mathcal{E}(s)$, there should be no magnetic field on axis, and Maxwell's equations should be satisfied in their homogenous

[1] This is a tautology, since AVF means azimuthally-varying field.

form (no charge and no current on axis). Based on Ref. [1] we propose to use:

$$A_x = A_z = 0\,, \tag{2}$$

$$A_s = -\mathcal{E}(s)\left(1 - \frac{\omega^2}{c^2}\frac{x^2(1+\iota)+z^2(1-\iota)}{4}\right)\frac{\sin(\omega t+\phi)}{\omega}\,, \tag{3}$$

$$\Phi = \mathcal{E}'(s)\cos(\omega t+\phi)\frac{x^2(1+\iota)+z^2(1-\iota)}{4}\,. \tag{4}$$

The "asymmetry" parameter ι can take any value between -1 and +1; $\iota = 0$ corresponds to the case of a cylindrical gap [1]; $\iota = -1$ corresponds to the case of a horizontal gap.

You can verify that, along the reference trajectory ($x = z = 0$), this set of potentials leads to the desired fields:

$$\vec{\mathcal{E}} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} = (0, 0, \mathcal{E}(s))\,, \tag{5}$$

$$\vec{B} = \nabla \times \vec{A} = (0, 0, 0)\,, \tag{6}$$

while at the same time satisfying Maxwell's equations in their homogenous form:

$$\nabla \cdot \vec{B} = 0\,, \tag{7}$$

$$\nabla \cdot \vec{\mathcal{E}} = 0\,, \tag{8}$$

$$\nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{\mathcal{E}}}{\partial t} = (0, 0, 0)\,, \tag{9}$$

$$\nabla \times \vec{\mathcal{E}} + \frac{\partial \vec{B}}{\partial t} = (0, 0, 0)\,. \tag{10}$$

Remember that the choice of potentials is not unique; suitable choices differ by a gauge transformation. Note that, with this particular choice of gauge, the electric potential Φ is zero on axis; The on-axis energy gain comes entirely from the time derivative of A_s.

From this Hamiltonian we readily obtain the longitudinal equations of motion for the reference particle ($x = z = 0$ and $P_x = P_z = 0$):

$$t' = -\frac{\partial \mathcal{H}}{\partial E} = \frac{E}{c\sqrt{E^2 - m^2c^4}} = \frac{1}{\beta c}\,, \tag{11}$$

$$E' = +\frac{\partial \mathcal{H}}{\partial t} = q\mathcal{E}\cos(\omega t + \phi)\,. \tag{12}$$

The signs come from the fact that the time coordinate t is canonically conjugate with $-E$. Note that the definition of phase that we use here leads to an energy gain per unit length proportional to the **cosine** of the phase. Some authors prefer to use the other convention, where it is proportional to the **sine** of the phase: beware!

To study the transverse motion close to the optical axis, we Taylor expand the Hamiltonian to second order in x, z, P_x, and P_z, leading to:

$$\mathcal{H}_2(x, P_x, z, P_z; s) = \frac{P_x^2}{2P} + \frac{P_z^2}{2P} + \frac{q}{2}\left(\frac{\mathcal{E}'C}{\beta c} - \frac{\mathcal{E}S\omega}{c^2}\right)\frac{x^2(1+\iota)+z^2(1-\iota)}{2}\,, \tag{13}$$

where $C = \cos(\omega t + \phi)$, $S = \sin(\omega t + \phi)$, and $P = \sqrt{\frac{E^2}{c^2} - m^2c^2}$ is the reference particle's momentum. This expanded Hamiltonian contains no first order term: it implies that a particle placed on the reference axis will remain on it. You may wonder what happened to the 0^{th} order term: we simply swept it under the rug, as it does not contribute to the dynamics.

For any given on-axis electric field $\mathcal{E}(s)$[2], the linear transport can be calculated numerically, to arbitrary precision, by integrating the equations of motions that derive from this Hamiltonian [1]. Let's however trade a little of this precision for some more intuitive comprehension.

0.2 Approximate Thin-Lens Formula

From the expanded Hamiltonian we obtain the following linear equation of motion:

$$P_z' = -\frac{\partial \mathcal{H}_2}{\partial z} = z\, q \frac{1-\iota}{2c} \left(\mathcal{E} S \frac{\omega}{c} - \frac{\mathcal{E}' C}{\beta} \right) . \tag{14}$$

Note that z can be swapped for x by changing the sign in front of the asymetry parameter ι. Integrating this equation for an incident parallel ray – neglecting the change in z during the crossing of the gap as in Ref. [3] – leads to:

$$\Delta P_z = z\, q \frac{1-\iota}{2c} \int_{-\infty}^{\infty} \left(\mathcal{E} S \frac{\omega}{c} - \frac{\mathcal{E}' C}{\beta} \right) \mathrm{d}s\, . \tag{15}$$

We take the second term first, noting that:

$$\frac{\mathcal{E}' C}{\beta} \mathrm{d}s = \mathrm{d} \left(\frac{\mathcal{E} C}{\beta} \right) - \mathcal{E} \left(\frac{C}{\beta} \right)' \mathrm{d}s, \tag{16}$$

and that the first term in this expression integrates to zero as $\mathcal{E} = 0$ at both ends of the integral. Further, $(C/\beta)' = C'/\beta - C\beta'/\beta^2$, $C' = -\omega S/(\beta c)$. The result is

$$\int \frac{\mathcal{E}' C}{\beta} \mathrm{d}s = \frac{\omega}{c} \int \frac{\mathcal{E} S}{\beta^2} \mathrm{d}s + \int \frac{\beta'}{\beta^2} \mathcal{E} C \mathrm{d}s. \tag{17}$$

We can now identify the first term in eqn. 15 as $-\beta^2$ times the first term here in eqn. 17, and thus see that the former is the effect of the rf magnetic field. Ordinarily in this non-relativistic regime for cyclotron injection, we could ignore it, but for completenes, we leave it in for now. The sum of the first term in eqn. 15 and the first term in eqn. 17 contributes a factor $\frac{1}{\beta^2} - 1 = \frac{1}{\beta^2\gamma^2}$, so

$$\int \left(\mathcal{E} S \frac{\omega}{c} - \frac{\mathcal{E}' C}{\beta} \right) \mathrm{d}s = -\frac{\omega}{c} \int \frac{\mathcal{E} S}{\beta^2\gamma^2} \mathrm{d}s - \int \frac{\beta'}{\beta^2} \mathcal{E} C \mathrm{d}s. \tag{18}$$

Using $\beta' = \frac{\gamma'}{\beta\gamma^3} = \frac{q\mathcal{E}C}{\beta m c^2}$, this becomes:

$$-\int \left(\mathcal{E} S \frac{\omega}{c} - \frac{\mathcal{E}' C}{\beta} \right) \mathrm{d}s = \frac{\omega}{c} \int \frac{\mathcal{E} S}{\beta^2\gamma^2} \mathrm{d}s + \frac{q}{mc^2} \int \frac{(\mathcal{E} C)^2}{\beta^3\gamma^3} \mathrm{d}s. \tag{19}$$

These integrals can now be approximately evaluated in the limit of short rf gaps of effective length $L_{\text{eff}} := \frac{V_{\text{g}}}{\mathcal{E}(0)}$ with a voltage V_{g} across them, giving a vertical momentum change:

$$\Delta P_z = -z \frac{1-\iota}{2c} \left[\frac{\omega}{c} \frac{q V_{\text{g}} \sin\phi}{\beta^2\gamma^2} + \frac{(q V_{\text{g}} \cos\phi)^2}{L_{\text{eff}} m c^2 \beta^3\gamma^3} \right] + \mathcal{O} \left(\frac{\Delta(\beta\gamma)}{\beta\gamma} \right)^2 . \tag{20}$$

[2]Not any: $\mathcal{E}(s)$ must be at least differentiable.

Because of the trapezoid rule used to approximate the integrals, the value of $\beta\gamma$ is the average between the value before and after crossing the gap. The higher order terms in $\frac{\Delta(\beta\gamma)}{\beta\gamma}$ are negligible when the relative momentum gain across the gap is $\ll 1$.

The term proportional to $(qV_\mathrm{g}\cos\phi)^2$ represents the "electrostatic focusing"in [4, sections 7]. It is identical to that produced by a DC gap of the same voltage. The other term is the pure rf part, including the effect from the electric and magnetic time varying fields.

For the cyclotron application, $\omega = \beta ch/R$ where h is the harmonic number, i.e., the ratio of rf frequency to revolution frequency. Further, at injection $\beta \ll 1$ and we set $\gamma = 1$. Let there be n_g rf gaps per turn; we assume for simplicity that the bunch phase ϕ is the same at each gap, though this can be generalized. The $\Delta\theta = 2\pi/n_\mathrm{g}$. Except for the very first rf gap, which is often close to being round ($\iota = 0$), the rest of the rf system is flat and focuses only in the z direction ($\iota = -1$). The smoothed betatron motion equation $z'' + \nu_z^2 z = 0$ gives $\nu_z^2 z = -\frac{R}{P}\frac{\mathrm{d}P_z}{\mathrm{d}\theta}$. Eqn. 20 can thus be transformed into an equation for vertical tune:

$$\nu_z^2 = -\frac{R}{zP}\frac{\Delta P_z}{\Delta\theta} = \frac{n_\mathrm{g}}{4\pi}\left[h\left(\frac{qV_\mathrm{g}\sin\phi}{E_\mathrm{k}}\right) + \frac{R}{2L_\mathrm{eff}}\left(\frac{qV_\mathrm{g}\cos\phi}{E_\mathrm{k}}\right)^2\right]. \tag{21}$$

E_k is the kinetic energy.

References

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