Beam Physics Note
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In this Order: Orbits, Tunes, and Magnet Design

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Abstract: In this note I brush up a little the theory that I had presented in my Cyclotrons'19 paper, to reflect the latest development in the `from_orbit` code. Then I present the case of a cyclotron magnet that I have designed using `OPERA-3D` from the isochronous field distribution obtained using the python package `from_orbit`.

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1 Introduction

In[1] I showed how to calculate the transverse tunes from the geometry of the closed orbits in a fixed field accelerator. In this paper, I did not make it very clear that the transverse tunes are entirely determined from geometry in the case of isochronous obits. It does not matter what type of particle you are dealing with, it is only the geometry of the orbits that matters for the tunes. Let me try to clarify this point in the next section.

In Section 3 I present an example of a cyclotron designed using the `from_orbit` code. In Section 4, I show how I used the finite element code `OPERA-3D` to determine a magnet geometry that produces the required field distribution.

2 Theory: Isochronous Orbits and Tunes

Let's recall the premisses. The first assumption is that the shape of all the closed orbits is known and given by:

$$r(a,\theta)\colon \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \,, \tag{1}$$

where r is the radius of the closed orbit, a is the orbit's average radius, and θ is the azimuth. The periodicity of the closed orbits imposes that:

$$r(a,\theta + 2\pi/N) = r(a,\theta + 2\pi/N) \text{ with } N \in \mathbb{N}^* \,, \tag{2}$$

where N is the lattice periodicity, i.e. number of sectors. Although is it not strictly necessary, we impose that:

$$\frac{\partial r}{\partial a} > 0 \,, \tag{3}$$

which guaranties that the closed orbit never cross. The last assumption is that the orbits are isochronous[1], which leads to the following relation between the particle velocity β (relative to the speed of light) and the orbit length $\mathcal{L}$:

$$\beta(a) = \frac{\mathcal{L}(a)}{2\pi \mathcal{R}_\infty} \,, \tag{4}$$

where $\mathcal{R}_\infty$ is a constant, $\mathcal{L}$ is the orbit circumference:

$$\mathcal{L}(a) = \int_0^{2\pi} \frac{\mathrm{d}s}{\mathrm{d}\theta}\, \mathrm{d}\theta \,, \tag{5}$$

and:

$$\frac{\mathrm{d}s}{\mathrm{d}\theta} = \sqrt{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2} \,. \tag{6}$$

So far, it is geometry, and only geometry.

Transverse tunes are obtained by numerically integrating:

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\theta} = \mathbf{X}'\frac{\mathrm{d}s}{\mathrm{d}\theta} = \mathbf{F}\mathbf{X}\frac{\mathrm{d}s}{\mathrm{d}\theta} \tag{7}$$

over one period for two different sets of initial transverse state vectors: $\mathbf{X} = (1,0,1,0,0,0)^\mathsf{T}$ and $\mathbf{X} = (0,1,0,1,0,0)^\mathsf{T}$, see for instance Ref. [2]. The infinitesimal transfer matrix $\mathbf{F}$ for a

[1] In[1], I treat the more general case of non-isochronous orbits, where $\beta(a)$ can be arbitrary, but let's not concern ourselves with that here.

fixed field magnet with mid-plane symmetry is given by[3]:

$$\mathbf{F} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{n-1}{\rho^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{n}{\rho^2} & 0 \end{pmatrix} . \tag{8}$$

One only needs to know how to calculate the field index n and the radius of curvature ρ at any point along an orbit to track the linear motion of particles around it, and obtain transverse tunes exact to arbitrary precision.

The orbit curvature is first obtained as in wikipedia:Curvature:

$$\frac{1}{\rho} = \frac{r^2 + 2\left(\frac{\partial r}{\partial \theta}\right)^2 - r\frac{\partial^2 r}{\partial \theta^2}}{\left(r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2\right)^{3/2}} . \tag{9}$$

The field index n is obtained from:

$$n = -\frac{\rho}{B_0}\frac{\partial B}{\partial x} = \frac{\partial \rho}{\partial x} - \frac{\rho}{\beta\gamma}\frac{\partial \beta\gamma}{\partial x} \tag{10}$$

where $\beta\gamma = \frac{\beta}{\sqrt{1-\beta^2}}$ and β is given by Eq. (4). As shown in [1]:

$$\frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial \rho}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{1}{r}\left(\frac{\partial \rho}{\partial a}\frac{\frac{\mathrm{d}s}{\mathrm{d}\theta}}{\frac{\partial r}{\partial a}} - \frac{\partial \rho}{\partial \theta}\frac{\frac{\partial r}{\partial \theta}}{\frac{\mathrm{d}s}{\mathrm{d}\theta}}\right) , \tag{11}$$

where $\frac{\mathrm{d}s}{\mathrm{d}\theta}$ is given by Eq. (6). If one chooses $\mathcal{R}_\infty$ to be the unit of length, the numerical integration is done entirely from the knowledge of $r(a,\theta)$ and its partial derivatives. And since the transverse tunes are unitless numbers, their values is not affected by the choice of the unit of length, i.e. independent of $\mathcal{R}_\infty$. The transverse tunes derive entirely from the shape of the orbits, and from absolutely nothing else.

Note that, to be able to calculate tunes this way, the function $r(a,\theta)$ must be sufficiently smooth for the following partial derivatives: $\frac{\partial r}{\partial \theta}, \frac{\partial r}{\partial^2 \theta^2}, \frac{\partial^3 r}{\partial \theta^3}, \frac{\partial r}{\partial a}, \frac{\partial^2 r}{\partial a \theta}, \frac{\partial^3 r}{\partial a \theta^2}$ to be defined.

3 Constant Tune Compact Cyclotron

The conventional approach to cyclotron design is to start from the geometry of the magnet, modeled in some 3-dimensional finite element code, and to modify this geometry to iteratively produce an isochronous field distribution. This typically takes many iterations of hours-long calculation to produce 1 field map. By contrast, when starting from some arbitrary function $r(a,\theta)$ one can produce an isochronous field map in a split second. One can then search the large parameter space of possible $r(a,\theta)$ functions to explore, relatively much more quickly, the range of possible isochronous distributions.

Let's consider the following way to parametrized the shape of the closed orbits:

$$r(a,\theta) = a\big(1 + C(a)\cos\left(N(\theta - \phi(a))\right)\big) , \tag{12}$$

where N is the number of sectors. One now needs to find a way to define the two functions $C(a)$ and $\phi(a)$ using a finite – and hopefully small – number of degrees of freedom. This can be done, for instance, by constraining the values of $C(a)$ and $\phi(a)$ for a finite number of orbits, and use a cubic spline interpolation to evaluate there function for any intermediate value of a.

In the example presented in Table 1, I define $r(a, \theta)$ following Eq. (12) constraining the values of $C(a)$ and $\phi(a)$ for only 5 different orbits. Since I am considering the design of a compact cyclotron, where the radius of the innermost orbit is small compared to the magnetic gap of the cyclotron magnet, the innermost orbit is necessarily very close to a perfect circle. For this reason I start with $C(0.004 \times \mathcal{R}_\infty) = 0$. The initial value of $\phi(a)$ can be chosen arbitrarilyL I chose $\phi(0.004 \times \mathcal{R}_\infty) = 0$. Four more values of C and ϕ remain to be chosen: the magnetic field of the entire cyclotron is parametrized with only $4 \times 2 = 8$ degrees of freedom.

To setup an optimization problem, one needs to choose an objective: I choose to minimize the RMS variation of the radial and vertical tunes of the acceleration range of the cyclotron. Now it is just a matter to let some optimization routine (I used python scipy.optimize.minimize) to adjust the 8 degrees of freedom to best satisfy the objective function. The result obtained in the case of a 3-sector cyclotron are presented in Table 1 and Figs. 1 to 3. Note that, because of the constraint that the innermost orbit should be circular, the tunes start around $\nu_r = 1$ and $\nu_z = 0$, but rapidly increase to $\nu_r \approx 1.3$ and $\nu_z = 0.47$ and remain there for the entire range of the machine (until $a = 0.425\mathcal{R}_\infty$).

$a/\mathcal{R}_\infty$	C	ϕ/rad
0.004	0.	0.
0.14	0.07297	0.5636
0.24	0.08343	0.5687
0.35	0.07129	0.4080
0.425	0.04683	0.2383

Table 1: Examples of orbit shape parameters. The names in the first row refer to the titles of the corresponding subsections.

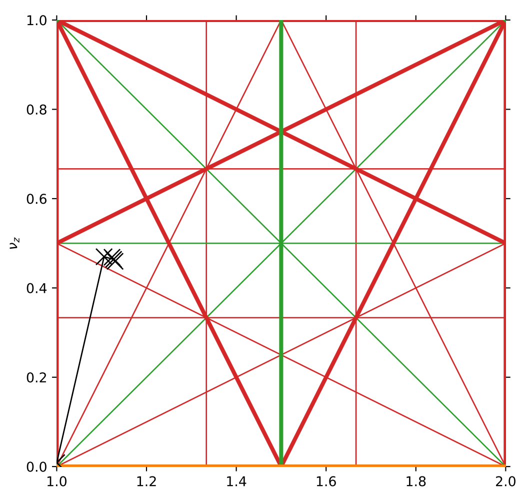

Figure 1: Tunes obtained using the parameters in Table 1. Betatron resonance conditions are shown up to 3[rd] order, with the structural resonances shown with thick lines and non-structural with thin lines.

To go any further with the study, we need to choose the parameters that will turn this purely geometrical problem into an actual cyclotron design: these are the value of $\mathcal{R}_\infty$ and the mass and charge of the particles to accelerate. Because I am interested in the design of a cyclotron such as this [4]), I choose $\mathcal{R}_\infty = 5\,\mathrm{m}$, and the particles to be H_2^+ ions. With this choice, one can now produce a field map of the cyclotron median plane from $r(a,\theta)$ using:

$$B(r,\theta) = \frac{\beta(a(r,\theta))}{\sqrt{1-\beta^2(a(r,\theta))}} \frac{m}{q\rho(a(r,\theta),\theta)}\,, \tag{13}$$

Note that the "inverse" function $a(r,\theta)$ must be constructed from $r(a,\theta)$ using some numerical method (I use a bicubic spline to construct this inverse function).

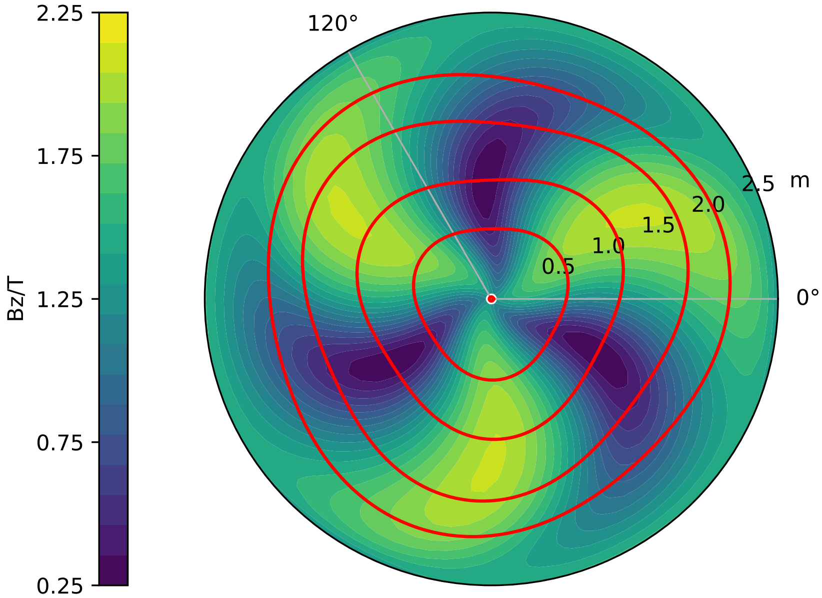

Figure 2: Example of isochronous field map produce using `from_orbit`. The 5 orbits shown in red were used to construct the spline function $r(a,\theta)$, which in turn was used to produce the magnetic field map.

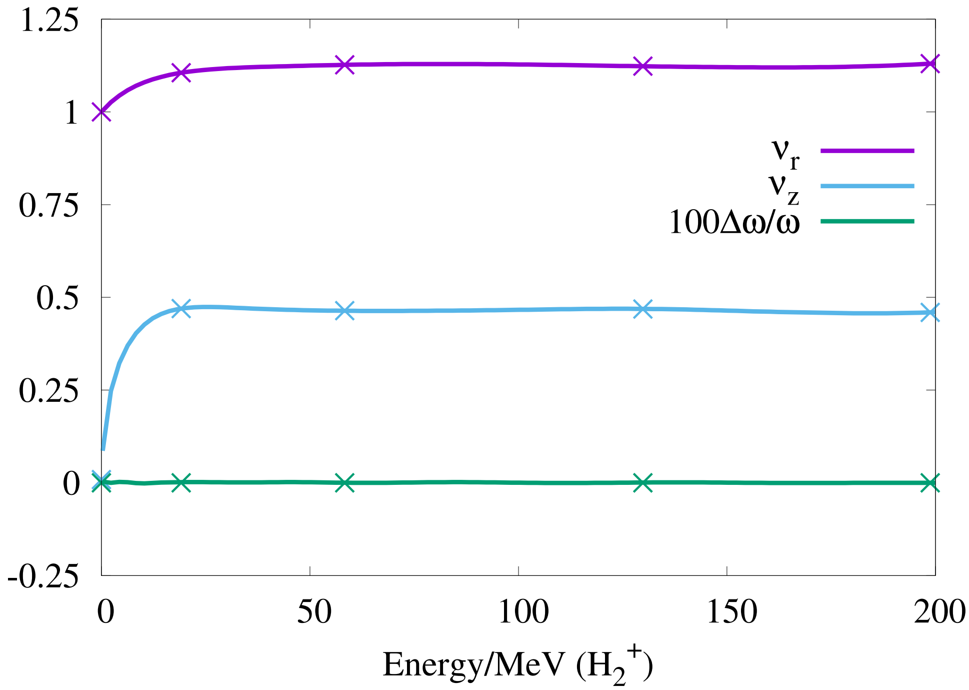

Figure 3: Transverse tunes, and relative variation of the orbital frequency, plotted as a function of the H_2^+ beam energy for the example shown in Fig. 2. Crosses are results of calculation from `from_orbit`. Solid line are results obtained after extracting a magnetic field map from `from_orbit`, and running it through the standard orbit code `CYCLOPS`.

4 Magnet Design

To design a magnet that would produce the same field distribution as in Fig. 2, I wrote a python script which call OPERA-3D iteratively, adjusting the shape of the pole face to produce the desired field distribution. The result is presented in Figs. 4 to 7.

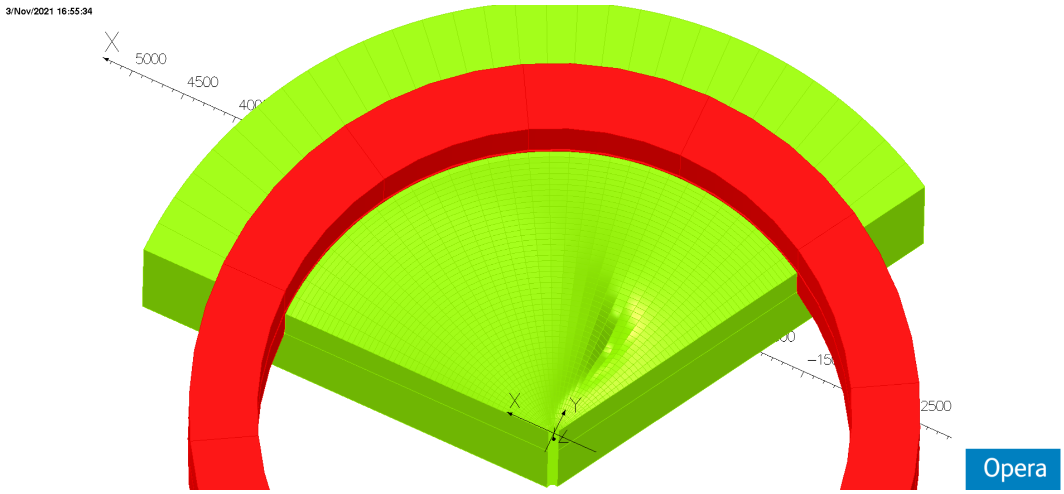

Figure 4: View of the OPERA-3D magnet model.

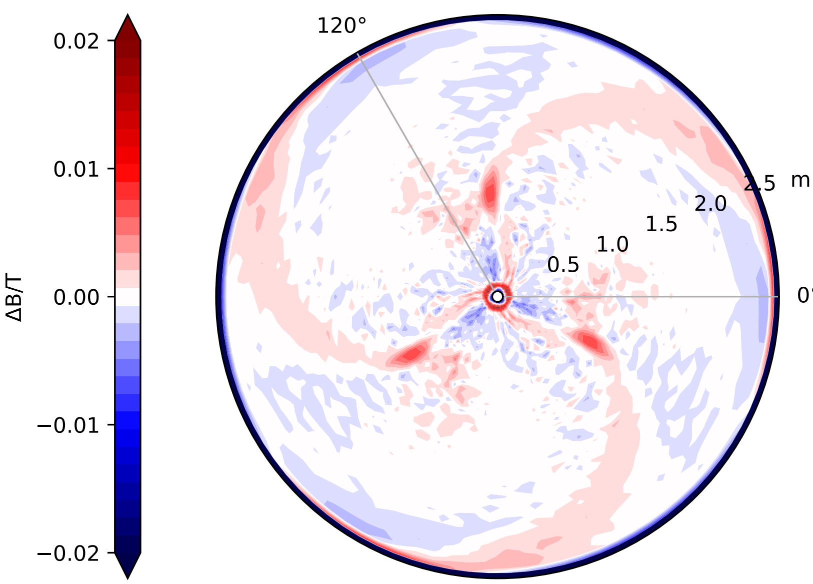

Figure 5: Difference between the desired and the achieved magnetic field in the magnet mid-plane.

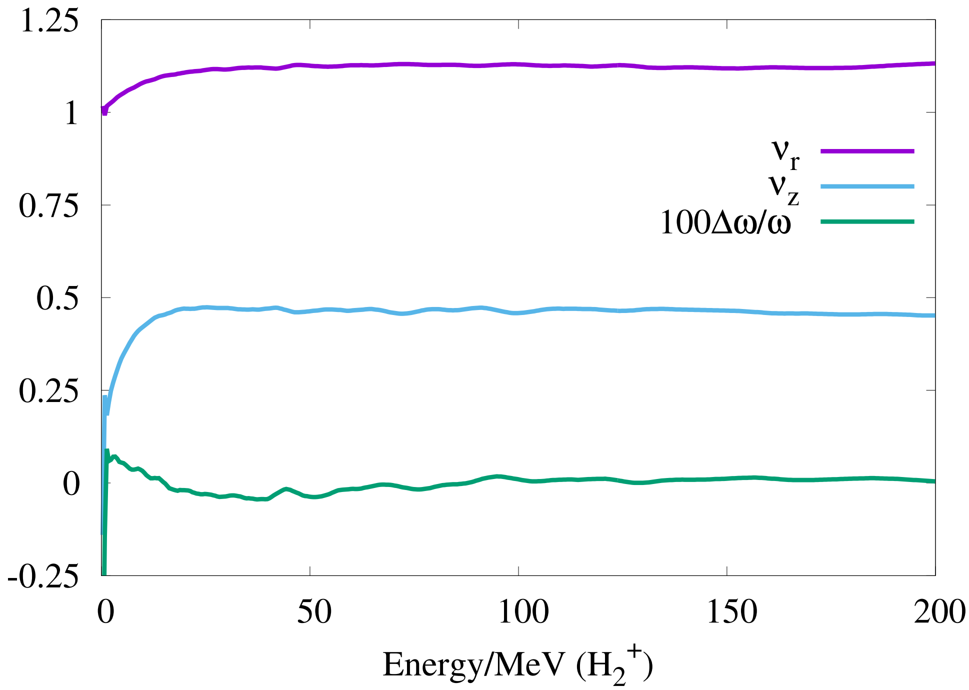

Figure 6: Transverse tunes, and relative variation of the orbital frequency, plotted as a function of the H_2^+ beam energy for the from a `CYCLOPS` simulation in the magnetic field map obtained from `OPERA-3d`.

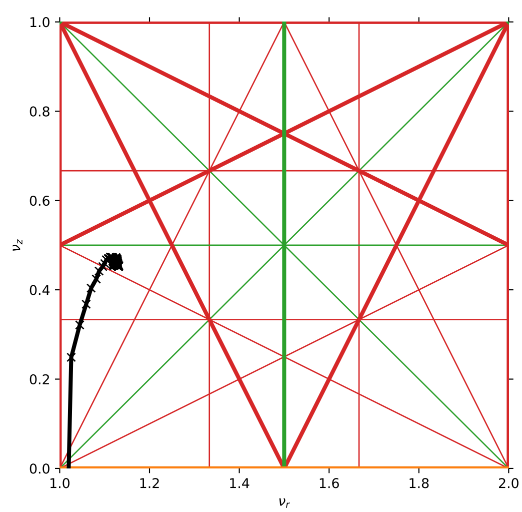

Figure 7: Same tunes as in Fig. 6, but plotted in the tune diagram, showing structural (thick lines) and non-structural (thin lines) resonance lines up to 3$^{\rm rd}$ order.

References

[1] T. Planche, Designing cyclotrons and fixed field accelerators from their orbits, in: Proc. Cyclotrons'19, 2019, pp. 353–357.

[2] C. Meade, Changes to the main magnet code, Tech. Rep. TRI-DN-70-54-Addendum4, TRIUMF (1971).

[3] R. Baartman, Linearized Equations of Motion in Magnet with Median Plane Symmetry, Tech. Rep. TRI-DN-05-06, TRIUMF (2005).

[4] Y.-N. Rao, R. Baartman, Y. Bylinskii, T. Planche, L. Zhang, et al., Conceptual design of TR100+: An innovative superconducting cyclotron for commercial isotopes production, in: 22nd Int. Conf. on Cyclotrons and their Applications (Cyclotrons' 19), Cape Town, South Africa, 23-27 September 2019, JACOW Publishing, Geneva, Switzerland, 2020, pp. 298–301.