Beam Physics Note
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Field From a Quadrupole With a Hyperbolic Tangent Strength Function

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Abstract: The 3-dimensional field distribution from a soft-edged quadrupole can always be obtain from a truncated Taylor series. However, in cases where the beam is large and occupies most of the quadrupole aperture, it is not obvious at what order to truncate the Taylor series. In such cases, it is desirable to have at one's disposal an exact solution to calculate the field from a soft-edges quadrupole in 3-d space. In this note I review the known solution from quads with a sech^2 strength function and I propose a extended solution for quads with fringe field fall-off given by a first-order Enge (i.e. tanh) strength function.

1 Sech Quadrupole

Derevjankin [1, 2] proposed a formula that gives the scalar potential for a quadrupole with arbitrary on-axis field gradient profile $k(z)$:

$$V(x,y,z) = -\operatorname{Re}\left\{\int_{z+iy}^{z+ix}\int k(\zeta)\mathrm{d}\zeta\mathrm{d}t\right\}. \tag{1}$$

Baartman used this formula to obtain an explicit expression for the potential from a quadrupole with $k(z) = \frac{K}{2}\operatorname{sech}^2 z$. The equation given in [3, Eq. (9)] can alternatively be written as:

$$V(x,y,z) = -\frac{K}{4}\log\left(\frac{\cos 2x + \cosh 2z}{\cos 2y + \cosh 2z}\right). \tag{2}$$

One can verify that, as expected, $\nabla^2 V = 0$. The explicit expression for $\vec{F} = \nabla V$ given in [3, Eq. (10–12)] does not need to be reproduced here. The electric field distribution from an electrostatic quadrupole is given by:

$$\vec{\mathcal{E}} = -\vec{F}, \tag{3}$$

where K is in unit of electric field.

The same formula can be used to derive the magnetic field distribution from a magnetic quadrupole after rotating the scalar potential by 45 degree:

$$\vec{B}(x,y,z) = -\nabla V\left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}, z\right), \tag{4}$$

provided that K is in unit of magnetic field. The application of the chain rule, and the fact that $\nabla\left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}, z\right)$ is the matrix of rotation around z by an angle of 45 degree, lead to:

$$\vec{B}(x,y,z) = -\vec{F}\left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}, z\right)\cdot\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{5}$$

2 Tanh Quadrupole

For quadrupoles that don't have a sech^2-like strength function, one can try and use the more general Enge function [4] to fit the field fall-off on each side of the magnet. Here we consider only the case of a quadrupole with identical entrance and exit edges, separated by a distance L and with all Enge coefficients equal to 0 except for $c_1 = 2/\lambda$. The strength function of our quadrupole becomes:

$$k(z) = \frac{K}{L}\left(\frac{1}{1+\exp 2\left(\frac{z-L/2}{\lambda}\right)} - \frac{1}{1+\exp 2\left(\frac{z+L/2}{\lambda}\right)}\right). \tag{6}$$

Note that $\int_{-\infty}^{\infty} k(z)\mathrm{d}z = K$, the integrated strength of our quadrupole. To keep the math nice and tidy we choose hereunder our unit of length so that $\lambda = 1$. Using the identity $\tanh x = 1 - \frac{2}{1+e^{2x}}$ we re-write:

$$k(z) = \frac{K}{2L}\left(\tanh(z+L/2) - \tanh(z-L/2)\right). \tag{7}$$

Using Eq. (1) leads to:

$$V = \frac{K}{4L}\,\mathrm{Re}\Big\{\mathrm{Li}_2\left(-e^{L+2(ix+z)}\right) - \mathrm{Li}_2\left(-e^{-L+2(ix+z)}\right) + \mathrm{Li}_2\left(-e^{-L+2(iy+z)}\right) - \mathrm{Li}_2\left(-e^{-+2(iy+z)}\right)\Big\} \tag{8}$$

where $\mathrm{Li}_2\,(t) = -\int_0^t \frac{\log 1-z}{z}\mathrm{d}z$ is the polylogarithm function of order 2. Admittedly, this expression would not be very usefull except for the fact that its gradient depends only on conventional trigonometric functions:

$$\begin{aligned} F_x &= \frac{K}{2L}\arctan\left(\frac{\sinh L \sin 2x}{\cosh L \cos 2x + \cosh 2z}\right) \\ F_y &= -\frac{K}{2L}\arctan\left(\frac{\sinh L \sin 2y}{\cosh L \cos 2y + \cosh 2z}\right) \\ F_z &= \frac{K}{4L}\log\left(\frac{(\cosh(L-2z)+\cos 2x)(\cosh(L+2z)+\cos 2y)}{(\cosh(L+2z)+\cos 2x)(\cosh(L-2z)+\cos 2y)}\right). \end{aligned} \tag{9}$$

Here again you can use Eq. (3) or (5) to obtain the corresponding electric or magnetic field distribution.

3 Example Use With Short Permanent Magnet Quads

TRIUMF recently received a shipment of permanent magnet quadrupoles, each with a length of 9.3 cm and integrated field strenght of 0.3 T. We performed a magnetic mapping to verify the properties of these quadrupoles. The measured gradient as a function of longitudinal distance is shown in figure 1. The integrated gradient obtained from this measurement is 0.338 T, which is approximately 12.7% higher than the initial magnet specifications. This plot shows that the sech^2 description does not match the permanent magnet gradient behaviour, motivating the use of the more general Tanh quadrupole description. Using equation Eq. (7) to fit this data, we obtain the second curve shown in figure 2, with fit parameters indicated in the legend.

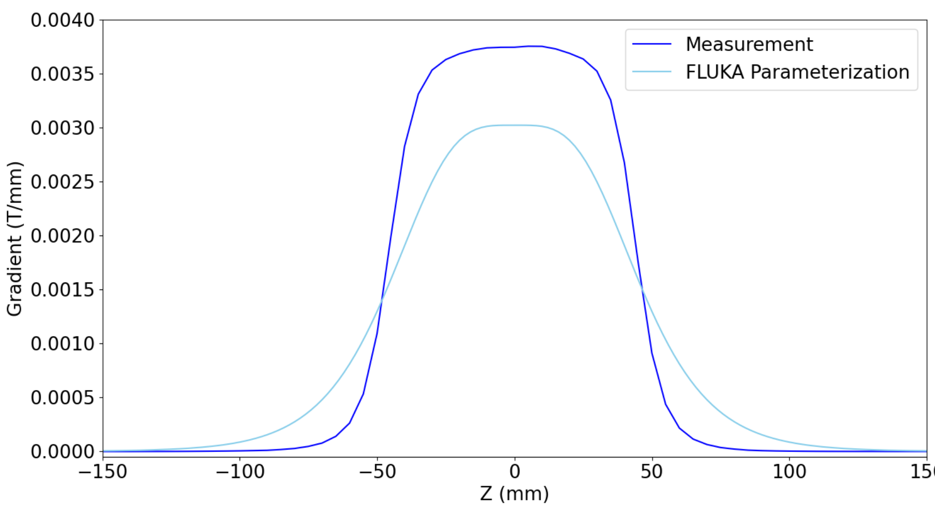

Figure 1: Measured gradient of 0.3 T permanent magnet quadrupole with previus sech^2 parameterization for comparison

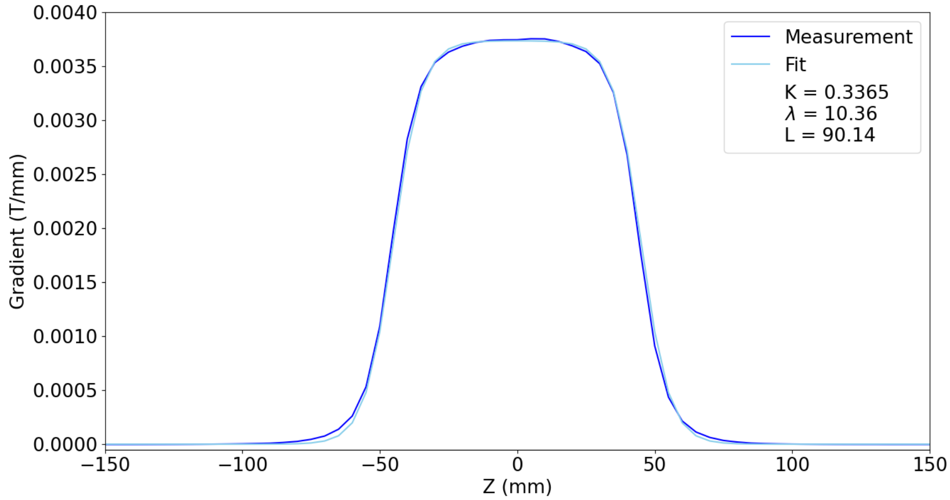

Figure 2: Measured gradient of 0.3 T permanent magnet quadrupole with tanh fit for comparison.

References

[1] G. Derevjankin, On the representation of the quadrupole lens potential, Zh. Tkh. Fiz.(USSR) 42 (6) (1972) 1178–1181.

[2] G. Vasil'ev, Spatial field of a quadrupole lens: An analytical approach, Nuclear Instruments and Methods 151 (1-2) (1978) 65–68.

[3] R. Baartman, Quadrupole shapes, Physical Review Special Topics-Accelerators and Beams 15 (7) (2012) 074002.

[4] H. A. Enge, Effect of extended fringing fields on ion-focusing properties of deflecting magnets, Review of Scientific Instruments 35 (3) (1964) 278–287.