**Beam Physics Note**
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# Stray Magnetic Field in TRANSOPTR

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**Abstract:** In this note we review the implementation of the longitudinal stray magnetic field in the envelope code TRANSOPTR. We also use the opportunity to add to the code the possibility to track the effect of the other two components of the stray field. In other words: we have added to TRANSOPTR the possibility to track the effect of an on-axis transverse dipolar magnetic field. We present a couple of test runs, including a simulation of the effect of the stray magnetic field in the new horizontal injection line of TRIUMF's 500 MeV cyclotron, which is currently being designed.

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# 1 Vector Potential

Let's consider a straight section of beamline: our reference trajectory is a straight line, no curvature. The Courant-Snyder Hamiltonian [1] reduces to:

$$H = -P_s = -qA_s - \sqrt{\left(\frac{(E - q\Phi)}{c}\right)^2 - m^2c^2 - (P_x - qA_x)^2 - (P_y - qA_y)^2}\,, \tag{1}$$

where $m$ and $q$ are the mass and charge of the particle; $(x, y, s, t)$ are the particle coordinates: transverse ($\times 2$), longitudinal, and time, respectively; the canonically conjugated momenta are $(P_x, P_y, P_s, -E)$; finally $\mathbf{A} = (A_x, A_y, A_s)$ is the magnetic vector potential and $\Phi$ is the electric scalar potential, all of which are assumed to be static here.

Let's find a suitable vector potential to describe our problem. What we want is to prescribe some arbitrary magnetic field distribution $B_x(s)$, $B_y(s)$, $B_s(s)$ along the reference trajectory. To expand the field off axis, we need to find a truncated Taylor series in $x$ and $y$ that satisfies Maxwell's equations (in their homogeneous form) along the axis. We choose the the lowest-possible-order series, which corresponds to the lowest-order field harmonics: solenoidal in the longitudinal and dipolar in the transverse directions.

You can verify that:

$$\begin{aligned} A_x &= -\frac{yB_s}{2} - \frac{y^2B'_y}{2}\,, \\ A_y &= \frac{xB_s}{2} + \frac{x^2B'_x}{2}\,, \\ A_s &= yB_x - xB_y\,, \end{aligned} \tag{2}$$

satisfies $\boldsymbol{\nabla} \times \mathbf{A} = (B_x, B_y, B_s)$ and $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{A} = \mathbf{0}$ along the reference axis. Note that the $x^2$ and $y^2$ terms required to satisfy the latter, once plugged into the Hamiltonian, lead only to terms of order higher than 2. This means that they do not affect the linear beam dynamics.

# 2 Equations of Motion in TRANSOPTR

For reasons of math simplicity, aesthetic of the equations in canonical form, and ease of debugging, it is important that when momentum is changing, only canonical coordinates are integrated. That would require to integrate $(x, P_x, y, P_y, t, E)$. But to relate to the standard optics convention of using $x'$,$y'$, and $\Delta p/p$, at every integration step we display these conventional quantities by dividing momenta by reference momentum. However, to keep it simpler yet, with dimensionless momenta, at every element, the momenta are in units of the reference momentum at the **start** of the integration step (which we call $P_\mathrm{i}$ hereunder). This requires rescaling those momenta after every step.

More explicitly, the corresponding Hamiltonian is obtain by first proceeding to the canonical transformation $(t,\, -E) \to (z,\, \Delta P) = (\beta_0 c\Delta t,\, \Delta E/(\beta_0 c))$ using a generating function of the second kind:

$$F_2(t, \Delta P) = \left(\int \frac{\mathrm{d}s}{\beta_0 c} - t\right)(E_0 + \beta_0 c\Delta P)\,, \tag{3}$$

and then by normalizing all the momenta (including the Hamiltonian itself) by $P_\mathrm{i}$. The resulting momenta are:

$$\begin{aligned} p_x &= P_x/P_\mathrm{i}\,, \\ p_y &= P_y/P_\mathrm{i}\,, \\ p_z &= \Delta P/P_\mathrm{i}\,, \end{aligned} \tag{4}$$

and the resulting Hamiltonian $\frac{H}{P_\mathrm{i}}$, once expanded to second order, becomes:

$$h = \frac{1}{2P_\mathrm{i}}\left(2q(xB_y - yB_x) + \frac{q^2B_s^2}{4P_0}(x^2+y^2) + qB_s(p_xy - p_yx) + P_0(p_x^2+p_y^2) + \frac{P_0}{\gamma_0^2}p_z^2\right) \quad (5)$$

where $\gamma_0^2 = \frac{1}{1-\beta_0^2}$ and $P_0 = \beta_0\gamma_0 mc$. Note that I got rid of the constant terms in the Hamiltonian, since they do not contribute to the dynamics. Note also that I have omitted the terms containing $\Phi$ and its derivative as their contribution can be (and is) added separately to the $\mathbf{F}$ matrix in the code. The stray magnetic field may indeed be superimposed over regions where electric fields are present, and so $\beta_0$ is in general not constant: $\beta_0(s)$, likewise for $P_0(s)$ but not for $P_\mathrm{i}$, which is a constant across every step of the numerical integration.

The equations of motion of the beam first and second moments, that `TRANSOPTR` integrates using standard fourth-order Runge-Kutta, are:

$$\begin{aligned}\frac{\mathrm{d}\langle\mathbf{X}\rangle}{\mathrm{d}s} &= \mathbf{F}_\perp + \mathbf{F}_\mathrm{e}\langle\mathbf{X}\rangle\,,\\ \frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}s} &= \mathbf{F}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{F}^\mathsf{T}\,,\end{aligned} \quad (6)$$

where

$$\mathbf{X} = (x, p_x, y, p_y, z, p_z)^\mathsf{T} \quad (7)$$

is the state vector of a single particle,

$$\langle\mathbf{X}\rangle = \frac{1}{N}\sum_{i=1}^{N}\mathbf{X} \quad (8)$$

is the vector of the beam's first moments ($N$ is the total number of particles),

$$\boldsymbol{\sigma} = \frac{1}{N}\sum_{i=1}^{N}(\mathbf{X} - \langle\mathbf{X}\rangle)(\mathbf{X} - \langle\mathbf{X}\rangle)^\mathsf{T} \quad (9)$$

is the $6\times6$ covariance matrix of the beam, which contains the second moments of the beam distribution. $\mathbf{F} = \mathbf{F}_\mathrm{e} + \mathbf{F}_\mathrm{sc}$ is the infinitesimal transfer matrix [2] and $\mathbf{F}_\perp(s)$ is a vector of the on-axis force: the product between the symplectic matrix and the gradient of the Hamiltonian evaluated on axis $\mathbf{X} = \mathbf{0}$.

In the case case we are considering in this note, where the dynamics is described by the Hamiltonian in Eq. (5), we can write explicitly:

$$\mathbf{F}_\perp(s) = \frac{q}{P_\mathrm{i}}\begin{pmatrix} -B_y(s) \\ 0 \\ B_x(s) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (10)$$

$$\mathbf{F}_{\mathrm{e}}(s) = \begin{pmatrix} 0 & \frac{P_{\mathrm{i}}}{P_0(s)} & \frac{qB_s(s)}{2P_0(s)} & 0 & 0 & 0 \\ -\frac{q^2B_s^2(s)}{4P_0(s)P_{\mathrm{i}}} & 0 & 0 & \frac{qB_s(s)}{2P_0(s)} & 0 & 0 \\ -\frac{qB_s(s)}{2P_0(s)} & 0 & 0 & \frac{P_{\mathrm{i}}}{P_0(s)} & 0 & 0 \\ 0 & -\frac{qB_s(s)}{2P_0(s)} & -\frac{q^2B_s^2(s)}{4P_0(s)P_{\mathrm{i}}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{P_{\mathrm{i}}}{\gamma^2P_0(s)} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{11}$$

and $\mathbf{F}_{\mathrm{sc}}$ is no different than for any other element in `TRANSOPTR` [3, 4]. Note that the factor $\frac{P_{\mathrm{i}}}{P_0(s)}$, which gives the relative change of momentum of the reference particle from the start of the element, corresponds to the internal variable “pratio” in the source code.

# 3 Testing With Localized Field Bumps

Tracking a 30 keV $^8Li^+$ through a field bump with an integral of $10^{-3}$ T m gives a total deflection of beam centroid of 14.16 mrad, see Fig. 1. This is consistent with the beam rigidity $B\rho$ of 0.07063 T m.

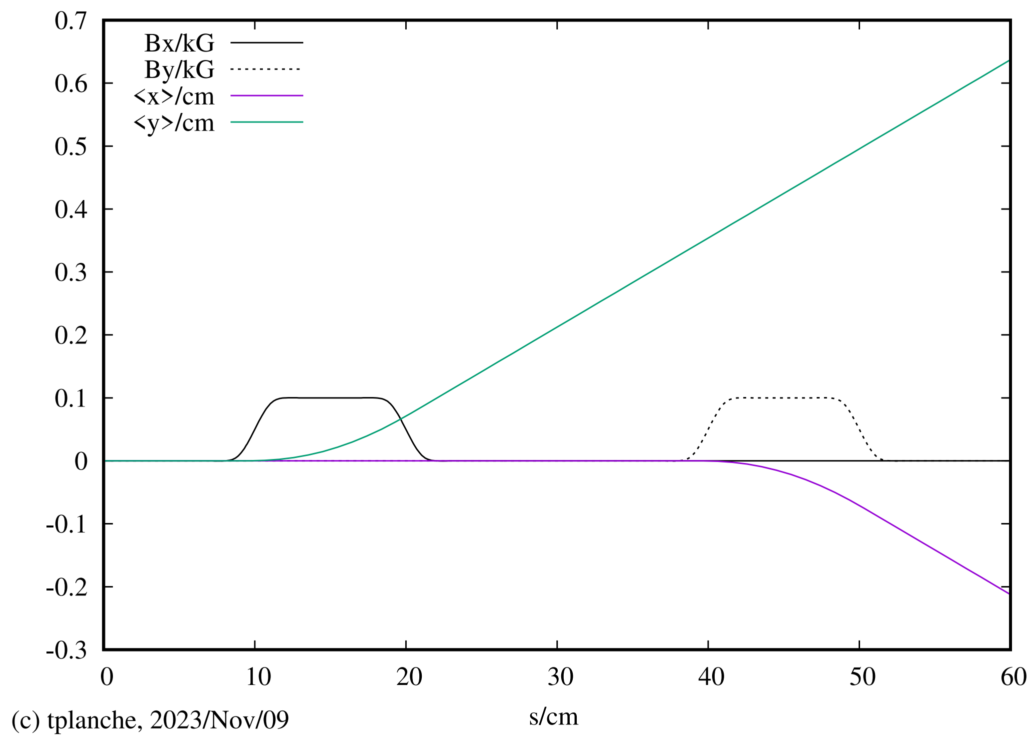


Figure 1: Deflection of a 30 keV $^8Li^+$ beam calculated by `TRANSOPTR` using the new “strayB” routine.

# 4 Testing on the Cyclotron Horizontal Injection Line

The phase advance per meter in the periodic section of the new horizontal injection line is about $k \approx 0.84$/m, together with average stray field of 0.12G ($\rho \approx 6500$ m) predicts average orbit error $\delta x = 1/(k^2\rho) \approx 0.22$mm. This agrees with Fig. 2.

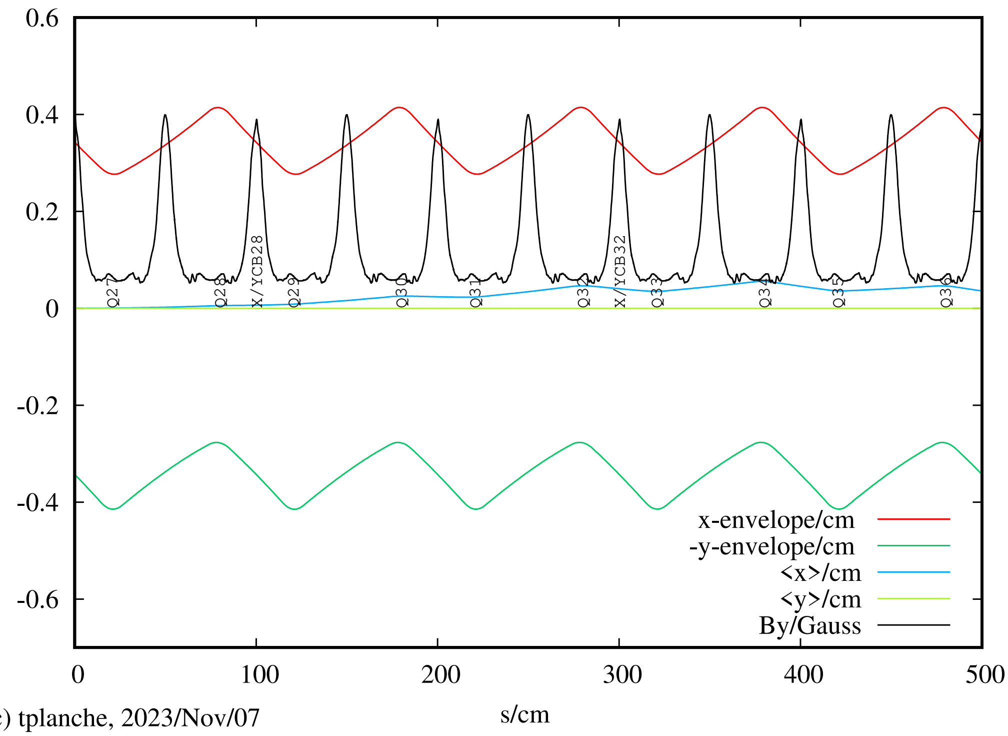


Figure 2: Evolution of the transverse beam size and centroid position through a periodic section of the new ISIS 300 keV $H^-$ injection line, currently under design at TRIUMF. The black line shows an estimate of the residual on-axis stray magnetic field which penetrates the prototype mu-metal shields: this data is actually comining from an `OPERA-3D` calculation of the effect of the shields.

# 5 Historical Note From Rick

When I [Rick] came to TRIUMF [1980], I was given the "theoretical" ISIS tune and it had the periodic section quads at 1.5kV. That would mean the phase advance per cell is only half what it is at 3 kV (something like 20 degrees). But they had 3 kV power supplies, and always were running them at full scale. I asked why if theory was 1.5 kV, why were they running at 3 kV? I finally got an answer from Mike Craddock, that less focusing results in larger beam size, and therefore smaller space charge force. That's what they wanted. But they got this scaling entirely wrong. The space charge tune shift is SMALLER when beam is focused to a smaller size. This is because space charge is $\propto 1/a$ while external force is $\propto 1/a^2$, where $a$ is beam radius.

They did not have our ability to calculate the central trajectory in the beamline, so possibly did not realize they would have needed strong focusing just to keep the beam close tp axis. Consider that their average field was greater than 1 Gauss [5, 6], so say that makes delta $\delta x$ 10 times larger, then with $k$ being 2 times smaller, that's a factor of 40 larger for $\delta x$, making it about 1 cm. That's why they needed 3 kV periodic voltage.

# References

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[3] R. Baartman, et al., Bunch dynamics through accelerator column, Proceedings of IPAC (2011).

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[5] R. Riches, Magnetic Shielding and Permanent Magnet Compensation as of November 1, 1975, Tech. Rep. TRI-DN-76-05, TRIUMF (1976).

[6] B. D. Remedios, ISIS Horizontal Beam Line Stray Magnetic Field Mapping and Compensation , Tech. Rep. TRI-BN-21-14, TRIUMF (2021).